TRINITY COLLEGE

Year 12 Mathematics Specialist Units 3, 4
Test 6 2021

Section 1 Calculator Free

Simple Harmonic Motion and Statistical Inference

STUDENT'S NAME Solutions

DATE: Tuesday 7 September **TIME:** 25 minutes **MARKS**: 22

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser, Formula Booklet

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (3 marks)

Determine an equation for y in terms of x for the second order differential equation $\frac{d^2y}{dx^2} = -2y$ given that $y'(0) = -\sqrt{2}$ and $y(0) = 0$

$$\frac{d^2y}{dx^2} = -(\sqrt{2})^2 y \Rightarrow y = A\sin(\sqrt{2}x + \alpha)$$

and $y' = \sqrt{2}A\cos(\sqrt{2}x + \alpha)$

Now $A\sin\alpha = 0$ —①

$\sqrt{2}A\cos\alpha = -\sqrt{2}$ —②

✓ SHM eqn

✓ y' and 2 eqns

So, ① ÷ ②

$\frac{1}{\sqrt{2}}\tan\alpha = 0 \Rightarrow \alpha = 0$

$\Rightarrow A = -1$

✓ solve and final eqn

$\therefore y = -\sin\sqrt{2}x$

2. (5 marks)

A particle is in simple harmonic motion. When it is at a distance d from the origin, its velocity is v, and when distant $2d$, its velocity is $\frac{1}{2}v$. Show that the period and the amplitude of the motion are $\frac{4\pi d}{v}$ and $d\sqrt{5}$ respectively.

SHM, so $x(t) = A\sin(kt+\alpha)$

Now $v^2 = k^2(A^2 - x^2)$

So $\left.\begin{aligned} v^2 &= k^2(A^2-d^2) \\ \left(\tfrac{1}{2}v\right)^2 &= k^2\left(A^2-(2d)^2\right) \end{aligned}\right\}$ ✓ 2 eqns

$\Rightarrow \left.\begin{aligned} v^2 &= k^2(A^2-d^2) \\ \frac{v^2}{4} &= k^2(A^2-4d^2) \end{aligned}\right\}$

Equating $\frac{v^2}{k^2}$

$\Rightarrow A^2 - d^2 = 4(A^2 - 4d^2)$ ✓ equate

$15d^2 = 3A^2$

$\therefore A = \sqrt{5}d$ ✓ show A

Now $v^2 = k^2(5d^2 - d^2)$ ✓ substitute to find k

$\Rightarrow \frac{v}{2d} = k$

So period, $T = \frac{2\pi}{v/2d}$ ✓ use Period to show T

$= \frac{4\pi d}{v}$

3. (7 marks)

A swinging pendulum is 5 metres long. Its horizontal displacement, $x(t)$ metres, at time t seconds can be **approximated** by the formula

$$x(t) = R\cos(\omega t + \alpha)$$

where R, ω, are positive constants, and $0 \le \alpha \le 2\pi$.

(a) Show that $x(t)$ undergoes simple harmonic motion. [2]

$x(t) = R\cos(\omega t + \alpha)$

$\dot{x}(t) = -\omega R\sin(\omega t + \alpha)$

$\ddot{x}(t) = -\omega^2 R\cos(\omega t + \alpha)$ ✓ second der

$= -\omega^2 x(t) \quad \therefore$ SHM ✓ $-\omega^2 x$

(b) Evaluate R, ω, and α, given that $x(0) = 2$, $\dfrac{dx}{dt}(0) = 0$, and $\dfrac{d^2x}{dt^2}(0) = -4$. [3]

$2 = R\cos(0 + \alpha)$ ①

$-4 = -\omega^2 R\cos(0 + \alpha)$ ③ ✓ 3 eqns

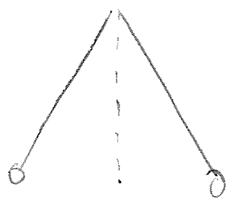
$0 = -\omega R\sin(0 + \alpha)$ ②

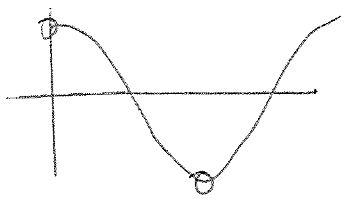
② ÷ ③ : $0 = \dfrac{\tan\alpha}{\omega} \Rightarrow \alpha = 0$ ✓ α

So $R = 2$ ✓ R and ω

$\omega = \sqrt{2}$

(c) How long does it take for the pendulum to swing from the furthest point on one side to the furthest point on the other? [2]





Period $T = \dfrac{2\pi}{\sqrt{2}}$

✓ Period

$x(t) = 2\cos(\sqrt{2}t)$

$\therefore$ time $= \dfrac{T}{2}$ ✓ Time

$= \dfrac{\pi}{\sqrt{2}}$ seconds

4. (7 marks)

A random sample of students at Trinity College found that the mean height of students was 165 cm. Repeated sampling of the mean indicated that the standard deviation of the sample means was 0.8 cm.

Given that $P(z > 1) = 0.1587$

$P(z > 1.5) = 0.0668$

$P(z > 2) = 0.0228$

Determine

(a) An approximate 86% confidence interval for the population mean height μ. [3]

$\therefore$ for approximation use $z = 1.5$ ✓ correct interval

So $165 - \frac{3}{2} \times 0.8 \leq \mu \leq 165 + \frac{3}{2} \times 0.8$ ✓ lower

$163.8 \leq \mu \leq 166.2$ ✓ upper

Another follow up random sample of one quarter the size of the original sample found that the mean height of students was 170 cm. Assume that both samples were drawn from the same population.

(b) Determine the standard deviation of the sample heights for the second sample. [2]

$\sigma(\bar{X}) = 0.8$

So $\frac{\sigma}{\sqrt{n}} = 0.8$

$\Rightarrow \sigma = 0.8\sqrt{n}$

For second sample

$\sigma(\bar{Y}) = \frac{0.8\sqrt{n}}{\sqrt{n/4}}$

$= 1.6$

✓ $\sigma(\bar{X})$

✓ $\sigma(\bar{Y})$

Suppose that 86% confidence intervals are calculated for each sample.

(c) Determine, with reasons, which confidence interval is more likely to contain the population mean. [2]

We can't tell. Even though the samples have different sizes, the C.I. are calculated based on these. The intervals either contain μ or they don't, but we cannot tell.

✓ statement

✓ reason

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Year 12 Mathematics Specialist Units 3, 4
Test 6 2021

Section 2 Calculator Assumed
Simple Harmonic Motion and Statistical Inference

STUDENT'S NAME ______________________________

DATE: Tuesday 7 September **TIME:** 25 minutes **MARKS**: 23

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Special Items: Three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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5. (4 marks)

Mr Presser takes three different random samples and determines a 95% confidence interval of each:

Sample 1:	$65.2 \le \mu \le 68.2$	3
Sample 2:	$65.8 \le \mu \le 67.4$	1.6
Sample 3:	$64.0 \le \mu \le 66.4$	2.4

Determine, with reasons:

(a) the sample with the largest sample size. [2]

largest sample size has narrowest interval ✓ interval

∴ Sample 2 ✓ Sample 2

(b) the sample with the most likely of containing the population mean μ. [2]

We can't tell. Each sample will either contain μ or not.

✓ Statement

✓ Reason

6. (10 marks)

Western Bus-a-lot wants to estimate the population mean travel time to work. They take a random sample of 400 commuters and determine a 95% confidence interval of commute time, in minutes, as $36.892 \le \mu \le 38.108$

(a) Determine the sample mean for this sample of 400 commuters. [2]

$$\bar{x} = \frac{36.892 + 38.108}{2}$$ ✓ average

$$= 37.5$$ ✓ $\bar{x}$

(b) Calculate, to the nearest 0.01, the sample standard deviation for this sample of 400 commuters. [3]

95% C.I. $\Rightarrow$ $k = 1.96$ ✓ k

So $38.108 - 37.5 = k \times \frac{\sigma}{\sqrt{n}}$

✓ error

$$\Rightarrow \sigma = \frac{38.108 - 37.5}{1.96} \times \sqrt{400}$$

$$= 0.31 \times \sqrt{400}$$

$$= 6.2$$ ✓ σ

To decrease trip duration, a new overpass is installed to avoid the railway line. A second sample of 200 commuters was taken and it was found that the sample standard deviation of commuter time was 8 minutes and 30 seconds and the total combined travel time for the sample was 106 hours and 40 minutes.

Western Bus-a-lot claim '*the overpass has been a huge success – travel with us and you will arrive at the busport sooner!*'.

(c) Perform the necessary calculations to comment on the company's claims. [5]

$\bar{x} = \dfrac{106 \times 60 + 40}{200}$ ✓ $\bar{x}$

$= 32$

$s = 8.5$

$n = 200$

95% C.I. $30.822 \leq \mu \leq 33.178$ ✓✓ Intervals

99% C.I. $30.452 \leq \mu \leq 33.540$

Because the intervals between the two differnt samples do not overlap, they are statistically different.

✓ Statistically diff

Therefore, there has been a change.

<u>But</u> we can't say the overpass has caused this. There might be bias in the data. Further analysis is required (at least sample the same number of people).

✓ difference does not imply cause.

7. (9 marks)

The raw examination mark for Mathematics Specialist is a normal variable with mean of 58.5 percent with a standard deviation of 17.25 percent. A random sample of 50 student papers is selected.

(a) State the distribution of the sample mean mark per paper. Justify your answer. [3]

$50 > 30 \quad \therefore$ approx normal ✓ $n > 30$

$\bar{X} \sim N\left(58.5, \left(\frac{17.25}{\sqrt{50}}\right)^2\right)$ ✓ μ

✓ var

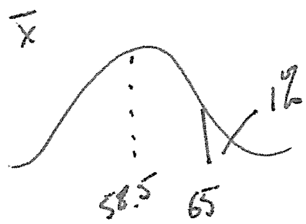
(b) Determine the probability that the sample mean mark is between 65 and 70 percent. [2]

$P(65 \le \bar{X} \le 70) = 0.00385$

✓✓

A different random sample of size n students was taken. Repeated sampling with this sample size shows that there is a 1% chance of obtaining a sample mean greater than 65%.

(c) Determine the value of n. [4]



So, by C.L.T.

$2.79414 = \frac{17.25}{\sqrt{n}}$

$\Rightarrow n = 38.11$

≈ 38

$P(z \geq k) = 0.01$

$\Rightarrow k = 2.3263$

Now $2.3263 = \frac{65 - 58.5}{\sigma_{\bar{x}}}$

$\sigma_{\bar{x}} = 2.79414$

✓ eqn

✓ k

✓ $\sigma_{\bar{x}}$

✓ ≈ 38